CSUC
Department of Physics
301A Mechanics
***PORTFOLIO* Problem 3c.)** due:

I. APPROXIMATION TO THE LEANING TOWER OF PISA EXPERIMENT: YOUR THIRD *PORTFOLIO* PROBLEM.

Introduction: Earlier in the semester you solved the famous Galileo "falling cannon ball" problem with an exact analytic solution (problem 2.36 on page 78 in our Text). This problem included a quadratic model for air resistance. You may recall that Galileo dropped both a musket ball and a much larger cannon ball at the same moment from the leaning tower of Pisa and then watched them hit the ground at almost the same moment. He stated that *"the larger outstrips the smaller by two finger-breadths"*. When you evaluated the solution, however, you found that it appears he wasn't being *"quite honest"*, though (let's be generous) ... maybe it all happened so fast he just had difficulty judging by eye! The difficulty with this exercise as you performed it, from a pedagogical point of view, is that we rarely (if ever !) get analytic solutions to our good physical models and, in this case, just the having of it allowed you to "breeze through" a lot of physical thinking that you really should have done. I'm guessing that you missed a lot (perhaps "most") of the physics and you missed the "oh-so-characteristic" experience that real physicists have to go through. You will come to really enjoy this *physical reasoning process* once you experience it more fully and internalize how it is done. It may seem strange to you that the procedure of the following PORTFOLIO problem which is to *skillfully* *approximate* this problem will teach you a whole lot more about how to do Physics than even finding the full solution ... but so it is. You will explore this scenario in four sets of questions presented below.

Let's get started by just noting some simple observations.

a) First, with no air resistance **all** objects fall identically and the two spheres would have hit the ground at the very same instant.

b) Second, since they didn't hit together some influence must have distinguished them. The key observation, though, is that they ***almost*** hit together! Whatever separated them was *"small"*.

c) In fact, we can check (and as part of this problem you will indeed check...) that the air resistance force given as:

$$F_{air\,resistance} \;=\; \kappa\,\rho_{air}\,A\,v^2 \tag{1}$$

really is very small compared to the gravitational force for the entire journey down to the ground. You may recall that κ is a dimensionless shape-factor having a value of very nearly $1/4$ for a sphere and that the term A is the cross sectional area of the sphere.

d) Once we are convinced that an added effect is *"small"*, we can include this small intrusion on our basic problem as a ***perturbation*** to be included in some consistent order. The added math is generally far simpler (and much more enlightening) than the formal solution itself.

1. Set up the zeroth order solution using natural units.

Suppose, as a first problem, that there is no air resistance ... and we lift a mass m up to a height h and then release it from rest. We know the answer to this problem, of course, but we wish to prepare the setting for our intended future perturbation analysis. Let us choose the origin of coordinates at the initial position and the downward direction as positive. This problem has only three natural sizes: $\{g,\, h,\, m\}$. In this initial "zeroth order" setup the mass drops out leaving only the initial height and the acceleration of gravity as natural sizes in the problem.

Question Set 1.) Use h as your unit of length L i.e. $h = 1 \cdot L$ and g as your unit of acceleration i.e. $g = 1 \cdot L\,/\,T^2$. First find the unit of time T and the unit of velocity L/T. Next, write down the ***Active*** equation of motion in these units and find the analytic solution. Now supposing we start our clock at the moment of release, please find the *actual pure numbers* representing the final quantities: $\{time \equiv t_{final},\, position \equiv x_{final},\, velocity \equiv v_{final}\}$ at the moment our mass impacts with the ground.

2. Now add in air resistance keeping these units.

Now we add in the air resistance. You may wish to start completely over from the familiar ***Passive*** statement of Newton's second law. However, make sure you now keep the same units as in the "zeroth order" problem above to arrive at our second ***Active*** equation of motion. Notice that all the remaining constants naturally ***cluster*** into a single **dimensionless** constant which serves as the *pre-factor* of the v^2 term. Let's designate this new "composite constant" by the Greek letter λ . Since the mass of the falling body is present in λ it will take on different values for the musket ball and the cannon ball, i.e. λ_{musket} and λ_{cannon}. However, the key insight is that this parameter may be viewed as a variable in its own right. It's not time dependent, of course, ... but as we set it at various values we are effectively describing different physical situations. If it were to take on the value zero, then we are back with the original "zeroth order" problem. As it pulls away from zero we may think of the air as becoming more and more significant. Since, in the units we're in, gravitational acceleration has the value unity and the final velocity is also of order "1" ... the importance of the air is solely determined by the size of this constant. So then, how big ***is*** this constant and, by the way, ... what does it represent? *Here comes the fun!* These dimensionless numbers can often be interpreted ***several*** significant ways, as you will next show.

Question Set 2.) Find the explicit (dimensionless) values of λ for both the musket ball and the cannon ball and verify that they are, indeed, *smallish* ... i.e. $\lambda_{m/c} \ll 1$. Then show that, up to simple dimensionless pre-factors of order "1", the "meaning" of λ could be any of: **a)** $(v_{final}/v_{terminal})^2$, **b)** (*energy lost to air resistance*)/(*initial energy*) , or ... the strangest one of all, **c)** (*mass of air pushed out of the way in falling*)/(*mass of the ball*).

In any case, this problem has a solution (you found it in that earlier problem set !) and that solution will be ***some*** smooth function of λ (let's call it $x(\lambda)$) and that function (whatever it may be) admits of a power series expansion in λ. That is, we will be able to write:

$$x(\lambda) \;=\; x_o \;+\; \lambda^1 \;\cdot\; x_1 \;+\; \lambda^2 \;\cdot\; x_2 \;+\; ... \tag{2}$$

And by taking time derivatives we find:

$$v(\lambda) \;=\; v_o \;+\; \lambda^1 \;\cdot\; v_1 \;+\; \lambda^2 \;\cdot\; v_2 \;+\; ... \tag{3}$$

The quantities $\{x_o,\ x_1,\ x_2\ ...\ \}$ and $\{v_o,\ v_1,\ v_2\ ...\ \}$ are functions of time representing the contributions toward the full solution of the parts proportional to ever higher powers of λ. In particular, $\{x_o\}$ should (and will) be the zeroth order solution obtained first above since it alone remains when λ goes to zero.

Now return to the result you achieved for the ***Active*** expression for Newton's second law for the full problem and insert the expansions in λ just above on both sides of the equal sign. By the uniqueness of power series we may equate the terms having equal powers of λ on either side. We now have an ***infinite*** set of equations to solve ! Each new equation will depend on the solution just previous to it ... so we must solve them sequentially. The point is, we happen to know that λ is small and only a few of these ***oh-so-simple*** equations will be necessary to get a really good approximation to the answer.

Question Set 3.) **a)** Find the zeroth and first order solutions explicitly. Then write down the approximation to the whole solution that is the sum of these two parts. Once you have done this, **b)** write down the difference in the solutions for the two falling balls and evaluate the difference between them $\Rightarrow$ ***consistent to first order in*** λ ***!*** $\Leftarrow$ when the first ball to hit the ground actually does so. Compare this with the full analytic solution obtained previously.

Question Set 4.) **a)** Now make your expression ***Passive*** once more by expressing your units in terms of the quantities they came from. Find the difference in position in explicit S.I. units. **b)** How does this difference scale with the height from which the ball were dropped ? **c)** How does this difference scale with the acceleration of gravity (e.g. how would it be different on the moon) ? **d)** How does this difference depend on the radii of the two balls?
e) To this order of approximation ... i.e. $\Rightarrow$ ***consistent to first order in*** λ ***!*** $\Leftarrow$ ..., what is the time difference in the hitting of the two balls? Is it plausible that Galileo might really have "missed" it ?